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A PLATFORMLESS INERTIAL NAVIGATION SYSTEM BASED ON A CANONICAL GRAVITATIONAL GRADIOMETER†

TSAI TIJING

China

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An algorithm is constructed for the ideal functioning of a platformless and gyroscope-free inertial navigation system which uses the readings from 18 accelerometers, which can be jointly considered as a gravitational gradiometer.

We consider a platformless inertial navigation system (PINS), the active elements of which are accelerometers and a canonical gravitational gradiometer (CGG). We introduce the following right systems of coordinates: (1) the Earth system of coordinates $O_1\xi\eta\zeta$, the origin of which is located at the centre of the Earth and the $O_1\zeta$ axis is directed along the vector of the absolute angular velocity of rotation of the Earth $\boldsymbol{\omega}_e$ and the $O_1\zeta$ axis directed along the Greenwich meridian; (2) a geocentric system of coordinates O_1xyz, the origin of which is located at the centre of the Earth, while the axes are parallel to the corresponding axes of the system of coordinates $Oxyz$ which is rigidly connected with the object and rotates with an absolute angular velocity $\boldsymbol{\omega}$. It is well known that the gravitational acceleration vector $\mathbf{g}(\mathbf{r})$ of the Earth's gravitation field is independent of the time t in the Earth system $O_1\xi\eta\zeta$. Hence, its total derivative with respect to time in the $O_1\xi\eta\zeta$ system is equal to

$$d\mathbf{g}/dt = \nabla\mathbf{g}\cdot\mathbf{v} = G(\mathbf{r})\cdot\mathbf{v}$$

where $G(\mathbf{r})$ is the tensor of the gradient of the acceleration due to gravity and $\mathbf{v}$ is the velocity vector of the object relative to the Earth. The absolute velocity $\mathbf{v}_a$ and the relative velocity $\mathbf{v}$ are related by the following equality

$$\mathbf{v}_a = \mathbf{v} + \boldsymbol{\omega}_e\times\mathbf{r}$$

where $\mathbf{r}$ is the radius vector from the centre of the Earth O_1 to the origin O of the system of coordinates $Oxyz$.

Calculating the absolute derivative with respect to time of both sides of the last equality, we obtain

$$d\mathbf{v}_a/dt = \dot{\mathbf{v}} + (\boldsymbol{\omega}+\boldsymbol{\omega}_e)\times\mathbf{v} + \boldsymbol{\omega}_e\times(\boldsymbol{\omega}_e\times\mathbf{r}) \tag{1}$$

where a derivative in the system of coordinates O_1xyz is denoted by a dot. By Newton's second law, the motion of an object satisfies the equation

$$d\mathbf{v}_a/dt = \mathbf{a} + \mathbf{g}' \tag{2}$$

where the apparent acceleration vector $\mathbf{a}$ is measured using accelerometers and $\mathbf{g}'$ is the vector of the gravitation field strength.

We introduce the gravitational gradient tensor $T(\mathbf{r})$, that is, $T(\mathbf{r}) = \nabla\mathbf{g}'$. Using Eq. (1), Eq. (2) can be written in the form

$$\dot{\mathbf{v}} + (\boldsymbol{\omega}+\boldsymbol{\omega}_e)\times\mathbf{v} = \mathbf{a}+\mathbf{g}, \quad \mathbf{g} = \mathbf{g}' - \boldsymbol{\omega}_e\times(\boldsymbol{\omega}_e\times\mathbf{r}) \tag{3}$$

On expressing the derivatives of the vectors $\mathbf{g}$ and $\mathbf{r}$ in the system $O_1\xi\eta\zeta$ in terms of the derivatives in the O_1xyz system, we obtain

$$\dot{\mathbf{g}} + (\boldsymbol{\omega}-\boldsymbol{\omega}_e)\times\mathbf{g} = G(\mathbf{r})\cdot\mathbf{v}, \quad \dot{\mathbf{r}} + (\boldsymbol{\omega}-\boldsymbol{\omega}_e)\times\mathbf{r} = \mathbf{v} \tag{4}$$

By calculating the time derivative in the system O_1xyz for both sides of Eq. (3) and taking account of Eq. (3) and the first equation of (4), we obtain

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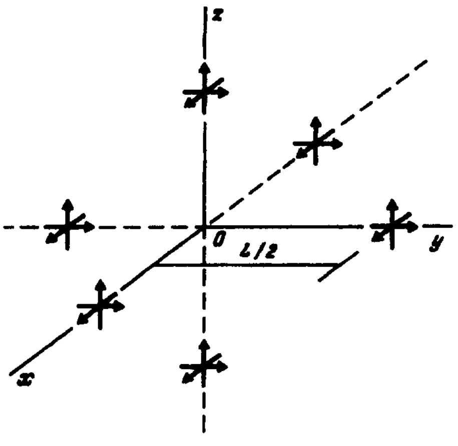

Fig. 1.

$$\ddot{\mathbf{v}}+2\Omega\cdot\dot{\mathbf{v}}+\left(\dot{\Omega}+\Omega^2-T(\mathbf{r})\right)\cdot\mathbf{v}=\dot{\mathbf{a}}+(\Omega-\Omega')\cdot\mathbf{a}$$

$$T(\mathbf{r})=G(\mathbf{r})+\Omega'^2 \tag{5}$$

$$\Omega=\begin{Vmatrix}0 & -\omega_z & \omega_y\\ \omega_z & 0 & -\omega_x\\ -\omega_y & \omega_x & 0\end{Vmatrix},\quad \Omega'=\begin{Vmatrix}0 & -\omega_{ez} & \omega_{ey}\\ \omega_{ez} & 0 & -\omega_{ex}\\ -\omega_{ey} & \omega_{ex} & 0\end{Vmatrix}$$

(the angular acceleration of the rotation of the Earth is equal to zero).

The CGG consists of six three-axis accelerometers or 18 single-axis high accuracy accelerometers located at a distance of $L/2$ from the origin of the system of coordinates $Oxyz$ (Fig. 1). The observed component of the gravitational gradient tensor is approximated by the finite difference between each pair of accelerometer readings which have the same axis of the system of coordinates.

For a uniaxial accelerometer, the Coriolis force is orthogonal to the velocity of the motion of the sensitive mass in the accelerometer. Consequently, it can be omitted. If force feedback is used in the accelerometer, we can assume that the relative acceleration of the sensitive mass is equal to zero. We therefore obtain

$$\mathbf{a}=\mathbf{a}_o+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})+\frac{d\boldsymbol{\omega}}{dt}\times\boldsymbol{\rho}-\mathbf{g}' \tag{6}$$

where $\boldsymbol{\rho}$ is the radius vector from the origin of the system of coordinates $Oxyz$ to the centre of the sensitive mass of the accelerometer, $\mathbf{a}_o$ is the absolute acceleration of the origin O of the system of coordinates $Oxyz$ and the acceleration $\mathbf{a}$ is measured by the accelerometer. If, apart from the gravity force, no other external forces of any kind act on the sensitive mass in the accelerometer, that is $\mathbf{a} = 0$, the relative acceleration is equal to the right-hand side of expression (6) with the opposite sign.

Using the readings from the accelerometers $\mathbf{a}_i$ at the points p_i ($i = 1, 2, \ldots, 6$), we obtain a formula from Eq. (6) for the change in the components of the gravitational gradient tensor

$$A=T-\Omega^2-\dot{\Omega} \tag{7}$$

where

$$T=\nabla\mathbf{g}_o,\quad A=\frac{1}{L}\begin{Vmatrix}(\mathbf{a}_2-\mathbf{a}_1)_x & (\mathbf{a}_2-\mathbf{a}_1)_y & (\mathbf{a}_2-\mathbf{a}_1)_z\\ (\mathbf{a}_4-\mathbf{a}_3)_x & (\mathbf{a}_4-\mathbf{a}_3)_y & (\mathbf{a}_4-\mathbf{a}_3)_z\\ (\mathbf{a}_6-\mathbf{a}_5)_x & (\mathbf{a}_6-\mathbf{a}_5)_y & (\mathbf{a}_6-\mathbf{a}_5)_z\end{Vmatrix} \tag{8}$$

Since $\dot{\Omega}$ is an antisymmetric tensor and T, Ω^2 are symmetric tensors, we obtain from Eq. (7) the angular acceleration

$$(2L)^{-1}\left[(\mathbf{a}_4-\mathbf{a}_3)_z-(\mathbf{a}_6-\mathbf{a}_5)_y\right]=\dot{\omega}_x\quad (135,\ 246,\ xyz) \tag{9}$$

(relations which have not been written out are obtained by cyclic permutation of the subscripts).

On integrating these equations, we obtain the absolute angular velocity $\boldsymbol{\omega}$.

We now consider the accuracy of the readings from the accelerometers which constitute the CGG. If an accuracy of 10^{-2}–10^{-4} E (Eötvös, 1E = $10^{-9}\,s^{-2}$) is required for the CGG and $L = 0.1$ m, it is necessary for the accelerometer to have a resolving power of 10^{-12}–10^{-14} m/s^2. If a resolving power with respect to the angular accelerometer in rotation of 10^{-11}–10^{-13} rad/s^2 is required then the accelerometer must have a resolving power of $10^{-12} \sim 10^{-14}$ m/s^2.

For a CGG based on Eq. (7), we can write Eq. (5) as

$$\ddot{\boldsymbol{v}} + 2\boldsymbol{\Omega}\cdot\dot{\boldsymbol{v}} - A\cdot\boldsymbol{v} = \dot{\mathbf{a}} + (\boldsymbol{\Omega}-\boldsymbol{\Omega}')\cdot\mathbf{a} \tag{10}$$

where the vectors $\mathbf{a}$ and $\boldsymbol{v}$ refer to the centre of the gradiometer O, and the components of the tensor A are measured by the CGG. On integrating Eq. (10) with respect to time and using Eq. (3), we obtain

$$\boldsymbol{v} = -2\int_0^t \boldsymbol{\Omega}\cdot \boldsymbol{v}d\tau + \int_0^t\int_0^t \left(2\dot{\boldsymbol{\Omega}} + A\right)\cdot \boldsymbol{v}d\tau d\tau + \int_0^t \mathbf{F}d\tau + \boldsymbol{v}^0$$

$$\mathbf{F} = \mathbf{a} + \int_0^t (\boldsymbol{\Omega}-\boldsymbol{\Omega}')\cdot \mathbf{a}d\tau + \mathbf{g}^0 - \left(\boldsymbol{\Omega}^0 - \boldsymbol{\Omega}'^0\right)\cdot \boldsymbol{v}^0 \tag{11}$$

An initial value is denoted by a zero superscript.

The angular acceleration vector is simultaneously measured by the CGG. On integrating Eq. (9), we obtain the angular velocity of rotation of the object

$$\omega_x = \frac{1}{2L}\int_0^t \left[\left(\mathbf{a}_4 - \mathbf{a}_3\right)_z - \left(\mathbf{a}_6 - \mathbf{a}_5\right)_y\right]d\tau + \omega_x^0 \quad (135,\ 246,\ xyz) \tag{12}$$

Since $\mathbf{a}$, $\boldsymbol{\omega}$, $\dot{\boldsymbol{\omega}}$ and A can be measured and calculated, the relative velocity $\boldsymbol{v}$ can first be found from Eq. (11) using the initial values of $\mathbf{r}^0$, $\boldsymbol{v}^0$, $\mathbf{g}^0$ and the specified angular velocity $\boldsymbol{\omega}_e$. Using the value which has been found, we then obtain the gravitational acceleration vector $\mathbf{g}$ and the position vector $\mathbf{r}$ from Eq. (3), the second equation of (4) and Eq. (11)

$$\mathbf{g} = -(\boldsymbol{\Omega}-\boldsymbol{\Omega}')\cdot \boldsymbol{v} + \int_0^t \left(2\dot{\boldsymbol{\Omega}} + A\right)\cdot \boldsymbol{v}d\tau + \mathbf{F} - \mathbf{a} \tag{13}$$

$$\mathbf{r} = \int_0^t \left(\boldsymbol{v} - (\boldsymbol{\Omega}-\boldsymbol{\Omega}')\cdot\mathbf{r}\right)d\tau + \mathbf{r}^0 \tag{14}$$

All the vectors $\mathbf{r}$, $\boldsymbol{v}$ and $\mathbf{g}$ are defined in the system of coordinates $O_1\xi\eta\zeta$.

In order to estimate the errors in a CGG-based PINS, we combine the equations for perturbed motion. On varying Eq. (3), the second equation of (4) and Eq. (10), we obtain

$$\begin{aligned}
&\delta\ddot{\boldsymbol{v}} + 2\boldsymbol{\Omega}\cdot\delta\dot{\boldsymbol{v}} - A\cdot\delta\boldsymbol{v} = \delta A\cdot\boldsymbol{v} - 2\delta\boldsymbol{\Omega}\cdot\dot{\boldsymbol{v}} + \delta\dot{\mathbf{a}} + (\boldsymbol{\Omega}-\boldsymbol{\Omega}')\cdot\delta\mathbf{a} + \delta\boldsymbol{\Omega}\cdot\mathbf{a}\\
&\delta\mathbf{g} = \delta\dot{\boldsymbol{v}} + (\delta\boldsymbol{\Omega} + \delta\boldsymbol{\Omega}')\cdot\boldsymbol{v} + (\boldsymbol{\Omega}+\boldsymbol{\Omega}')\cdot\delta\boldsymbol{v} - \delta\mathbf{a}\\
&\delta\dot{\mathbf{r}} + (\delta\boldsymbol{\Omega} - \delta\boldsymbol{\Omega}')\cdot\mathbf{r} + (\boldsymbol{\Omega}-\boldsymbol{\Omega}')\cdot\delta\mathbf{r} = \delta\boldsymbol{v}
\end{aligned} \tag{15}$$

Suppose $\delta\mathbf{a}$, $\delta\boldsymbol{\omega}$ and δA are the instrumental errors of the meter, the accelerators and the CGG, respectively. We denote the deviations of the position vector, the velocities and the gravitational acceleration vector from the values corresponding to the unperturbed operation of the system by $\delta\mathbf{r}$, $\delta\boldsymbol{v}$ and $\delta\mathbf{g}$.

In order to find the effect of instrumental errors of the inertial meters on the errors of the PINS, we will consider the case of the motion of an object along the equator at a constant velocity $\boldsymbol{v}_o$ when the Earth is a non-rotating sphere, that is, $\boldsymbol{\omega}_e = 0$. Then

$$T = \omega_0^2\,\mathrm{diag}(-1,\ -1, 2),\quad \mathbf{g} = [0, 0,\ -g]^T$$

$$\boldsymbol{\omega} = [0, \omega, 0]^T,\quad \boldsymbol{v} = [v_o, 0, 0]^T,\quad \mathbf{r} = [0, 0, R]^T$$

where $\omega_0 = \sqrt{(g/R)}$ is the Schuler frequency and R is the distance from the centre of the Earth to the point of the object.

We assume that $\boldsymbol{\omega}$ is small. We shall then neglect products of $\boldsymbol{\omega}$ and the variations $\delta\mathbf{r}$, $\delta\boldsymbol{v}$, $\delta\mathbf{g}$. Equations (15) can be written in scalar form in the following manner

$$\delta\ddot{v}_x + \omega_0^2\delta v_x = \delta A_{xx}v_o + \delta\omega_y g,\quad \delta\ddot{v}_y + \omega_0^2\delta v_y = \delta A_{yx}v_o - \delta\omega_x g$$

$$\delta\ddot{v}_z - 2\omega_0^2\delta v_z = \delta A_{zx}v_o$$

$$\delta g_x = \delta \dot{v}_x - \delta a_x, \quad \delta g_y = \delta \dot{v}_y + \delta\omega_z v_o - \delta a_y \tag{16}$$

$$\delta g_z = \delta \dot{v}_z - \delta\omega_y v_o - \delta a_z$$

$$\delta \dot{r}_x - \delta v_x = -\delta\omega_y R, \quad \delta \dot{r}_y - \delta v_y = \delta\omega_x R, \quad \delta \dot{r}_z - \delta v_z = 0$$

We will consider the instrumental errors of all the sensitive elements as mean values on the right-hand sides of this system of equations, that is, for constant $\delta\omega$, $\delta\mathbf{a}$, δA. We therefore assume that they do not change with time.

The solutions of this system of equations can be represented in the form

$$\frac{\delta r_x}{R} = \frac{\delta r_x^0}{R} + \frac{\delta A_{yx}}{\omega_0^2}\omega t + \xi_1 \sin\omega_0 t - \eta_1 \cos\omega_0 t$$

$$\frac{\delta r_y}{R} = \frac{\delta r_y^0}{R} + \frac{\delta A_{yx}}{\omega_0^2}\omega t + \xi_2 \sin\omega_0 t + \eta_2 \cos\omega_0 t$$

$$\frac{\delta r_z}{R} = \frac{\delta r_z^0}{R} - \frac{\delta A_{zx}}{2\omega_0^2}\omega t + \frac{1}{\sqrt{2}}\xi_3 \operatorname{sh}\left(\sqrt{2}\omega_0 t\right) + \frac{1}{2}\eta_3 \operatorname{ch}\left(\sqrt{2}\omega_0 t\right)$$

$$\frac{\delta v_x}{\omega_0 R} = \frac{\delta\omega_y}{\omega_0} + \frac{\delta A_{yx}}{\omega_0^2}\,\frac{\omega}{\omega_0} + \xi_1 \cos\omega_0 t + \eta_1 \sin\omega_0 t$$

$$\frac{\delta v_y}{\omega_0 R} = -\frac{\delta\omega_x}{\omega_0} + \frac{\delta A_{yx}}{\omega_0^2}\,\frac{\omega}{\omega_0} + \xi_2 \cos\omega_0 t - \eta_2 \sin\omega_0 t$$

$$\frac{\delta v_z}{\omega_0 R} = -\frac{\delta A_{zx}}{2\omega_0^2}\,\frac{\omega}{\omega_0} + \xi_3 \operatorname{ch}\left(\sqrt{2}\omega_0 t\right) + \frac{1}{\sqrt{2}}\eta_3 \operatorname{sh}\left(\sqrt{2}\omega_0 t\right)$$

$$\frac{\delta g_x}{g} = -\frac{\delta a_x}{g} - \xi_1 \sin\omega_0 t + \eta_1 \cos\omega_0 t$$

$$\frac{\delta g_y}{g} = \frac{\delta\omega_z}{\omega_0^2}\omega - \frac{\delta a_y}{g} - \xi_2 \sin\omega_0 t - \eta_2 \cos\omega_0 t$$

$$\frac{\delta g_z}{g} = -\frac{\delta\omega_y}{\omega_0^2}\omega - \frac{\delta a_z}{g} + \sqrt{2}\xi_3 \operatorname{sh}\left(\sqrt{2}\omega_0 t\right) + \eta_3 \operatorname{ch}\left(\sqrt{2}\omega_0 t\right)$$

$$\omega = u_0/R$$

$$\xi_1 = \frac{\delta v_x^0}{\omega_0 R} - \frac{\delta\omega_y}{\omega_0} - \frac{\delta A_{yx}}{\omega_0^2}\,\frac{\omega}{\omega_0}, \quad \xi_2 = \frac{\delta v_y^0}{\omega_0 R} + \frac{\delta\omega_x}{\omega_0} - \frac{\delta A_{yx}}{\omega_0^2}\,\frac{\omega}{\omega_0}, \quad \xi_3 = \frac{\delta v_z^0}{\omega_0 R} + \frac{\delta A_{zx}}{2\omega_0^2}\,\frac{\omega}{\omega_0}$$

$$\eta_1 = \frac{\delta g_x^0}{g} + \frac{\delta a_x}{g}, \quad \eta_2 = \frac{\delta\omega_z}{\omega_0^2}\omega - \frac{\delta g_y^0}{g} - \frac{\delta a_y}{g}, \quad \eta_3 = \frac{\delta g_z^0}{g} + \frac{\delta a_z}{g} + \frac{\delta\omega_y}{\omega_0^2}\omega$$

It can be see that, first, the error in the horizontal position of the object, to which the error of the gravitational gradiometer reduces, increases with time. Second, the error of the gravitational gradiometer δA reduces to the oscillatory component of the error in determining the horizontal component of the acceleration due to gravity, the velocity and position. Third, the errors in the position, velocity and acceleration due to gravity increase without limit with time.

Translated by E.L.S.